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Inequalities for Cyclic Functions

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The nth cyclic function is defined by

$$\varphi_n(z)=\sum_{\nu=0}^{\infty}\frac{z^{n\nu}}{(n\nu)!} \qquad (z\in\mathbb{C},\ 2\leqslant n\in\mathbb{N}).$$

We prove that if k is an integer with $1\leqslant k\leqslant n-1$, then

$$\left(\frac{(n-k)!\,\varphi_n^{(k)}(x)}{x^{n-k}}\right)^{\alpha}<\varphi_n(x)<\left(\frac{(n-k)!\,\varphi_n^{(k)}(x)}{x^{n-k}}\right)^{\beta}$$

holds for all positive real numbers x with the best possible constants

$$\alpha=1 \qquad\text{and}\qquad \beta=\binom{2n-k}{n}.$$

Key Words: inequalities; cyclic functions; gamma function.

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1. INTRODUCTION

The nth *cyclic function* φ_n $(2 \leqslant n \in \mathbb{N})$ is defined as the solution of the differential equation

$$y^{(n)}(z) = y(z), \qquad y(0) = 1, \qquad y'(0) = \cdots = y^{(n-1)}(0) = 0,$$

that is, as the entire function

$$\varphi_n(z) = \sum_{\nu=0}^{\infty} \frac{z^{n\nu}}{(n\nu)!} = \frac{1}{n} \sum_{\nu=0}^{n-1} \exp(\varepsilon_n^{\nu} z) \qquad (\varepsilon_n = e^{2\pi i/n}, \quad z \in \mathbb{C}).$$

For $n = 2$ we obtain the classical hyperbolic functions

$$\varphi_2(z) = \cosh z \qquad \text{and} \qquad \varphi_2'(z) = \sinh z.$$

The cyclic functions $\varphi_n, \varphi_n', \ldots, \varphi_n^{(n-1)}$ have found the attention of several authors who established many interesting properties of these functions; we refer to [1] and the references therein.

This paper has been inspired by the known inequalities

$$\text{(1.1)} \qquad \frac{\sinh x}{x} < \cosh x < \left(\frac{\sinh x}{x}\right)^3 \qquad (x > 0).$$

Simple proofs of (1.1) are given in [3] and [4]. Lazarević [3, p. 270] pointed out that in the second inequality of (1.1) the exponent 3 is best possible, that is, this inequality is not true for all $x > 0$, if we replace 3 by a smaller constant.

Double-inequality (1.1) leads to inequalities involving the arithmetic, geometric, and logarithmic means of two positive real numbers. Indeed, setting $x = \frac{1}{2} \log \frac{b}{a}$ $(b > a > 0)$ in (1.1), and multiplying by $\sqrt{ab}$, we obtain

$$L(a, b) < A(a, b) < (L(a, b))^3/(G(a, b))^2,$$

where $A(a, b) = \frac{1}{2}(a+b)$, $G(a, b) = \sqrt{ab}$, and $L(a, b) = \frac{a-b}{\log a - \log b}$. Related results can be found in [2, Chap. VI].

Páles [4, 5] provided other interesting applications of inequalities (1.1). He showed that they play a role in establishing inequalities for differences of powers and for two-parameter mean value families.

Since (1.1) can be written as

$$\text{(1.2)} \qquad \frac{1}{x} \varphi_2'(x) < \varphi_2(x) < \left(\frac{1}{x} \varphi_2'(x)\right)^3 \qquad (x > 0),$$

it is natural to look for an extension of (1.2) involving φ_n and its derivatives. It is the aim of this paper to present such an extension. More precisely, we answer the following question:

Let n and k be integers with $1 \leqslant k \leqslant n-1$. What is the largest number $\alpha = \alpha(n, k)$ and what is the smallest number $\beta = \beta(n, k)$ such that the inequalities

$$\left(\frac{(n-k)!\,\varphi_n^{(k)}(x)}{x^{n-k}}\right)^{\alpha} < \varphi_n(x) < \left(\frac{(n-k)!\,\varphi_n^{(k)}(x)}{x^{n-k}}\right)^{\beta} \tag{1.3}$$

hold for all positive real numbers x?

2. A DISCRETE INEQUALITY

In order to determine the best possible values α and β in (1.3) we need the following discrete inequality which might be of independent interest.

THEOREM 1. *Let u, v, x, and y be nonnegative integers such that $(u, v) \neq (0, 0)$ and $1 \leqslant y \leqslant x-1$, and let*

$$F(u, v, x, y) = \frac{(x+y)!}{(u+x+y)!\,v!} - \frac{x!\,y!}{(u+x)!\,(v+y)!}. \tag{2.1}$$

Then we have

$$(u+x)\,F(u, v, x, y) + (v+x)\,F(v, u, x, y) > 0. \tag{2.2}$$

Using Euler's gamma function to define

$$a! = \Gamma(a+1) \tag{2.3}$$

for nonnegative real numbers a, it is likely that inequality (2.2) holds for all real numbers $u, v, y \geqslant 0$ and $x \geqslant 1$. In fact, a detailed analysis of our proof of Theorem 1 reveals that (2.2) is valid at least for large parts of this latter range.

Proof. In what follows we use the abbreviations

$$H(u, v, x, y) = (u+x)\,F(u, v, x, y) + (v+x)\,F(v, u, x, y),$$

$$\gamma(s) = \Gamma(s+1), \qquad \text{and} \qquad \psi(s) = \Psi(s+1) = \frac{\Gamma'(s+1)}{\Gamma(s+1)} \qquad (0 \leqslant s \in \mathbb{R}),$$

and we employ freely the extension of $F(u, v, x, y)$ to nonnegative real numbers via (2.3). We have to show that

$$H(u, v, x, y) > 0. \tag{2.4}$$

Since $H(u, v, x, y)$ is symmetric in u and v, it suffices to establish (2.4) for $v \geqslant u$. We consider three cases.

Case 1: $u = v$. Since ψ' is strictly decreasing on $[0, \infty)$, we conclude from the integral representation

$$H(u, u, x, y)$$
$$= \frac{2(u+x)\,\gamma(x)\,\gamma(x+y)}{\gamma(u+x)\,\gamma(u+x+y)} \int_0^y \frac{\gamma(s)\,\gamma(u+x+s)}{\gamma(u+s)\,\gamma(x+s)} \int_s^{u+s} [\psi'(t) - \psi'(t+x)]\,dt\,ds$$

that

$$H(u, u, x, y) \geqslant 0, \tag{2.5}$$

with equality holding if and only if $u = 0$.

Case 2: $v \geqslant 2u+1$. Let

$$S(u, x, y) = \frac{(u+x-1)(x+y)!}{(u+x+y)!} + \frac{u!\,x!\,y!}{(u+x)!\,(u+y)!}.$$

Then we have

$$v!\,H(u, v, x, y) - u!\,F(u, u, x, y) \tag{2.6}$$
$$> S(u, x, y) - \frac{v!\,x!\,y!}{(v+y)!}\left[\frac{1}{(u+x-1)!} + \frac{(v+y)!}{(u+y)!\,(v+x-1)!}\right].$$

A simple calculation yields

$$S(u, x, y) \geqslant \frac{u!\,x!\,y!}{(u+x-1)!\,(u+y)!}, \tag{2.7}$$

so that (2.6) and (2.7) lead to

$$v!\,H(u, v, x, y) - u!\,F(u, u, x, y) \tag{2.8}$$
$$> x!\,y!\left[\frac{u!}{(u+x-1)!\,(u+y)!} - \frac{v!}{(v+y)!}\left(\frac{1}{(u+x-1)!} + \frac{(v+y)!}{(u+y)!\,(v+x-1)!}\right)\right].$$

Since $y \leqslant x-1$ and $2u+1 \leqslant v$, we get

$$\frac{(v+y)!}{(u+y)!\,(v+x-1)!} \leqslant \frac{1}{(u+x-1)!} \quad \text{and} \quad \frac{2v!}{(v+y)!} \leqslant \frac{u!}{(u+y)!}. \tag{2.9}$$

Using (2.9) we obtain

(2.10)

$$\frac{v!}{(v+y)!}\left(\frac{1}{(u+x-1)!}+\frac{(v+y)!}{(u+y)!\,(v+x-1)!}\right) \leqslant \frac{v!}{(v+y)!}\,\frac{2}{(u+x-1)!}$$
$$\leqslant \frac{u!}{(u+x-1)!\,(u+y)!}.$$

From (2.8), (2.10), and (2.5) we conclude

$$H(u, v, x, y) > \frac{u!}{v!}\,F(u, u, x, y) = \frac{u!}{2v!}\,\frac{H(u, u, x, y)}{u+x} \geqslant 0.$$

Case 3: $0 < u < v \leqslant 2u$. We define

$$G(t) = G_{u, v, x, y}(t) = \frac{\gamma(x+t)\,\gamma(y-t)}{\gamma(u+x+t)\,\gamma(v+y-t)} \qquad (-x \leqslant t \leqslant y).$$

Since

$$G_{v, u, x, y}(t) = G_{u, v, x, y}(y-x-t),$$

we have the representation

(2.11)

$$\begin{aligned} H(u, v, x, y) &= [(u+x)\,G_{u, v, x, y}(t) + (v+x)\,G_{v, u, x, y}(t)]_{t=0}^{t=y} \\ &= [(u+x)\,G_{u, v, x, y}(t) - (v+x)\,G_{u, v, x, y}(t-x)]_{t=0}^{t=y} \\ &= \left[\left[(u+v+x-s)\,G_{u, v, x, y}\left(t-\frac{v-s}{v-u}\,x\right)\right]_{s=u}^{s=v}\right]_{t=0}^{t=y} \\ &= \int_0^y \int_u^v \left\{-G'(\tau) + \left(\frac{x(u+x)}{v-u} + \frac{x(v-s)}{v-u}\right) G''(\tau)\right\} ds\, dt, \end{aligned}$$

where

$$\tau = t - \frac{(v-s)\,x}{v-u} \in [-x, y].$$

Direct computation yields

(2.12)
$$G''(t) = (X^2 + Y)\, G(t),$$

where

(2.13)
$$X = \frac{G'(t)}{G(t)}$$

and

(2.14)
$$Y = \left(\frac{G'(t)}{G(t)}\right)' = \psi'(x+t) - \psi'(x+t+u) + \psi'(y-t) - \psi'(y-t+v).$$

Since ψ' is decreasing, we conclude from (2.14) that $Y \geqslant 0$, so that (2.12) leads to

(2.15)
$$G''(t) \geqslant 0.$$

From (2.11) and (2.15) we get the estimate

$$H(u, v, x, y) \geqslant \int_0^y \int_u^v \left\{ -G'(\tau) + \frac{x(u+x)}{v-u}\, G''(\tau) \right\} ds\, dt.$$

Thus, to prove (2.4) it suffices to show that for $\tau \in [-x, y]$:

(2.16)
$$0 < -G'(\tau) + \frac{x(u+x)}{v-u}\, G''(\tau) = m(\tau), \text{ say.}$$

Let $A = \frac{x(u+x)}{v-u}$, and let X and Y be as given in (2.13) and (2.14) (with τ instead of t). Then we have

(2.17)
$$\begin{aligned} m(\tau)\, \frac{A}{G(\tau)} + \frac{1}{4} &= \left(AX - \frac{1}{2}\right)^2 + A^2 Y \\ &\geqslant A^2 Y \\ &= \left(\frac{x(u+x)}{v-u}\right)^2 [\psi'(x+\tau) - \psi'(x+\tau+u) + \psi'(y-\tau) - \psi'(y-\tau+v)] \\ &= P_{u,v,x,y}(\tau), \text{ say.} \end{aligned}$$

We have $u < v \leqslant 2u$ and $y < x$. Since ψ' is decreasing and ψ'' is increasing on $[0, \infty)$, we get

(2.18)

$$P_{u,v,x,y}(\tau) \geqslant \left(\frac{x(u+x)}{u}\right)^2 [\psi'(x+\tau)-\psi'(x+\tau+u)+\psi'(x-\tau)-\psi'(x-\tau+u)]$$
$$= Q_{u,x}(\tau), \text{ say.}$$

The function $\tau \mapsto Q_{u,x}(\tau)$ is even on $[-x, x]$, and since ψ''' is decreasing on $[0, \infty)$, we verify that it is also convex. Hence, it attains its minimum at $\tau = 0$. This leads to

$$(2.19)\qquad \begin{aligned} Q_{u,x}(\tau) &\geqslant Q_{u,x}(0) \\ &= 2\left(\frac{x(u+x)}{u}\right)^2 [\psi'(x)-\psi'(x+u)] \\ &= 2\left(\frac{x(u+x)}{u}\right)^2 \sum_{k=1}^{\infty} \left(\frac{1}{(x+k)^2} - \frac{1}{(x+k+u)^2}\right) \\ &\geqslant 2\left(\frac{x(u+x)}{u}\right)^2 \left(\frac{1}{(x+1)^2} - \frac{1}{(x+1+u)^2}\right) \\ &\geqslant 2\frac{x^2}{(x+1)^2} \\ &> \frac{1}{4}. \end{aligned}$$

From (2.17)–(2.19) we conclude the validity of (2.16) for $\tau \in [-x, y]$. This completes the proof of Theorem 1. ∎

3. MAIN RESULT

We are now in a position to prove the following extension of inequality (1.2).

THEOREM 2. *Let n and k be integers with* $1 \leqslant k \leqslant n-1$. *Then we have for all positive real numbers x:*

$$(3.1)\qquad \left(\frac{(n-k)!\,\varphi_n^{(k)}(x)}{x^{n-k}}\right)^{\alpha} < \varphi_n(x) < \left(\frac{(n-k)!\,\varphi_n^{(k)}(x)}{x^{n-k}}\right)^{\beta},$$

with the best possible constants

$$\alpha = 1 \qquad \text{and} \qquad \beta = \binom{2n-k}{n}.$$

Proof. The left-hand inequality of (3.1) with $\alpha = 1$ is an immediate consequence of the identity

$$\varphi_n(x) - \frac{(n-k)!\,\varphi_n^{(k)}(x)}{x^{n-k}} = (n-k)! \sum_{\nu=1}^{\infty} \left(\prod_{\mu=1}^{n-k} \left(1 + \frac{n\nu}{\mu}\right) - 1 \right) \frac{x^{n\nu}}{(n(\nu+1)-k)!}.$$

To prove the second inequality of (3.1) with $\beta = \binom{2n-k}{n}$, we define

$$f(x) = \beta \log(n-k)! + \beta \log \varphi_n^{(k)}(x) - \beta(n-k) \log x - \log \varphi_n(x).$$

We have to show that $f(x) > 0$ for $x > 0$. Since $\lim_{t \to 0} f(t) = 0$, it is sufficient to prove for $x > 0$:

$$f'(x) = \beta \frac{\varphi_n^{(k+1)}(x)}{\varphi_n^{(k)}(x)} - \frac{\beta(n-k)}{x} - \frac{\varphi_n'(x)}{\varphi_n(x)} > 0,$$

or, equivalently,

$$\text{(3.2)} \qquad \beta x \varphi_n^{(n)}(x)\,\varphi_n^{(k+1)}(x) - \beta(n-k)\,\varphi_n^{(n)}(x)\,\varphi_n^{(k)}(x) - x\varphi_n'(x)\,\varphi_n^{(k)}(x) > 0.$$

Expanding the expression on the left-hand side into a power series around 0, inequality (3.2) can be written as

$$\text{(3.3)} \qquad \sum_{m=3}^{\infty} \sum_{\nu=1}^{m-1} \frac{a_\nu + a_{m-\nu}}{2} x^{nm-k} > 0,$$

where

$$a_\nu = a_\nu(m, n, k) = \frac{n\nu\beta}{(n(m-\nu-1))!\,(n(\nu+1)-k)!} - \frac{1}{(n\nu-k)!\,(n(m-\nu)-1)!}.$$

If we set

$$u = n(\nu-1), \qquad v = n(m-\nu-1), \qquad \text{and} \qquad y = n-k,$$

then we obtain

$$\text{(3.4)} \qquad (u+n)\,F(u, v, n, y) + (v+n)\,F(v, u, n, y) = (a_\nu + a_{m-\nu})\, n!(n-k)!,$$

where F is given in (2.1). Since $m \geqslant 3$, $1 \leqslant v \leqslant m-1$, and $1 \leqslant k \leqslant n-1$ imply $u \geqslant 0$, $v \geqslant 0$, $(u, v) \neq (0, 0)$, and $1 \leqslant y \leqslant n-1$, we conclude from Theorem 1 and (3.4) that

$$a_v + a_{m-v} > 0 \qquad (v = 1, \ldots, m-1)$$

which implies (3.3).

It remains to show that the constants $\alpha = 1$ and $\beta = \binom{2n-k}{n}$ are best possible in (3.1). Let

$$g(x) = \frac{\log \varphi_n(x)}{\log \dfrac{(n-k)!\, \varphi_n^{(k)}(x)}{x^{n-k}}}; \tag{3.5}$$

then double-inequality (3.1) is equivalent to

$$\alpha < g(x) < \beta \qquad (x > 0). \tag{3.6}$$

We have

$$g(x) = \frac{\log(1+s(x))}{\log(1+t(x))},$$

with

$$s(x) = \sum_{v=1}^{\infty} \frac{x^{nv}}{(nv)!} \qquad \text{and} \qquad t(x) = (n-k)! \sum_{v=1}^{\infty} \frac{x^{nv}}{(nv+n-k)!}.$$

Hence, we obtain

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{s'(x)}{t'(x)} = \binom{2n-k}{n}. \tag{3.7}$$

If $1 \leqslant v \leqslant n-1$, then we have $\lim_{x \to \infty} \exp((\varepsilon_n^v - 1)\, x) = 0$. This implies

$$\lim_{x \to \infty} \frac{\varphi_n^{(r)}(x)}{\varphi_n^{(r-1)}(x)} = \lim_{x \to \infty} \frac{1 + \sum_{v=1}^{n-1} \exp((\varepsilon_n^v - 1)\, x)\, \varepsilon_n^{vr}}{1 + \sum_{v=1}^{n-1} \exp((\varepsilon_n^v - 1)\, x)\, \varepsilon_n^{v(r-1)}} = 1 \qquad (r = 1, \ldots, n),$$

which leads to

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{\varphi_n'(x)}{\varphi_n(x)} \frac{1}{\dfrac{\varphi_n^{(k+1)}(x)}{\varphi_n^{(k)}(x)} - \dfrac{n-k}{x}} = 1. \tag{3.8}$$

From (3.6)–(3.8) we conclude that in (3.1) the constants $\alpha = 1$ and $\beta = \binom{2n-k}{n}$ are sharp, which completes the proof of Theorem 2. ▮

Remark. Numerous computer calculations have led to the conjecture that the function g, as defined in (3.5), is strictly decreasing on $[0, \infty)$.

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